Toulouse School of Economics,
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J. Bolte

Problem Set

Tikhonov regularization method

Problem. Let $(H, \langle\cdot,\cdot\rangle)$ be a finite dimensional space and $f : H \to \mathbb{R}$ a differentiable convex function. For any integer $k \geq 1$, we define x_k to be a minimizer of $f(\cdot) + \frac{1}{2k}\|\cdot\|^2$. For short we write

$$x_k \in \text{argmin } \left\{ f(x) + \frac{1}{2k}\|x\|^2 : x \in H \right\}.$$

1/ Prove in a simple manner that f is bounded from below by an affine function.

2/ a/ Show that a continuous function $g : H \to \mathbb{R}$ such that $g(x) \to +\infty$ as $\|x\| \to +\infty$ has at least a minimizer.

b/ (*) Use **1/** to establish that the function $f_k(x) = f(x) + \frac{1}{2k}\|x\|^2$ satisfies $f_k(x) \to +\infty$ as $\|x\| \to +\infty$.

c/ Show that the sequence $(x_k)_{k\in\mathbb{N}}$ is well defined and unique.

3/ Let us assume that argmin $f \neq \varnothing$. Prove that $(x_k)_{k\in\mathbb{N}}$ is bounded.

4/ Prove that any convergent subsequence of $(x_k)_{k\in\mathbb{N}}$ has a its limit in argmin f.

5/ One considers the set $S = \big\{x \in \text{argmin } f \mid \|x\| \leq \|y\|,\ \forall y \in \text{argmin } f\big\}$. Show that S contains a unique element.

6/ Prove that x_k converges to the unique element of S.

7/ Let us consider now that $f(x) = \frac{1}{2}\|Ax - b\|^2$ where A is a matrix in $\mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $\|\cdot\|$ denote the usual euclidean norm on $\mathbb{R}^m$. Prove that f is convex, twice differentiable and has at least one minimizer.

8/ Let x^* be a minimizer of f. Prove that argmin $f = x^* + \ker A$.
Express x_k as a function of data's problem.
Explain briefly what you understand of the purpose of Tikhonov's method.

On convex cones

The space $\mathbb{R}^n$ is endowed with the Euclidean scalar product $\langle\cdot,\cdot\rangle$.

Exercise 1. We say that a closed convex cone is "self-dual" if $C^* = -C$.
Establish that the following sets are auto-dual cones. In each case one must verify carefully

that the set is actually a cone and sketch a graphical representation for $n = 2, 3$.
a) $C_1 = \mathbb{R}^n_+$ (nonnegative orthant).
b) $C_2 = \{x \in \mathbb{R}^n : x_1 \geq 0, x_1^2 \geq x_2^2 + \ldots + x_n^2\}$ (second-order cone).

Exercise 2. Compute the normal cone (at each point) of the following closed convex sets :
a) (*) $C_3 := \{x \in \mathbb{R}^n : ||x|| \leq 1\}$
b) (**) $C_4 := \{x \in \mathbb{R}^n : Ax \leq b\}$ $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. For this case, one may simply conjecture a formula for the normal cone, the proof is difficult and is thus optional.

Correction

Problem

1/ Take any point $\bar{x}$ and write the convex inequality at that point.

2/ a/ Let u_k be a sequence such that $g(u_k) \to \inf g$. If u_k was not bounded there would exist $\|u_{k_p}\| \to +\infty$ which would imply $g(u_{k_p}) \to +\infty$. Hence u_k is bounded and converges up to an extraction to a point $\bar{u}$. By continuity we have $g(\bar{u}) = \inf g$.

b/ The function f_k is continuous. By 1/ there exists $x^* \in \mathcal{E}$ and c such that

$$\begin{aligned} f_k(u) &\geq \langle x^*, u\rangle + c + \frac{1}{2k}\|u\|^2 \\ &\geq \frac{1}{2k}\|u\|^2 - \|x^*\|\|u\| + c. \end{aligned}$$

and the result follows from the fact that the limit of the polynomial $p(s) = \frac{1}{2k}s^2 - \|x^*\|s + c$ at $+\infty$ is $+\infty$.

c/ By the previous result f_k has a minimizer. Since f_k is also strictly convex as a sum of a strictly convex function $\frac{1}{2k}\|\cdot\|^2$ and a convex function, the minimizer is unique.

3/ Let a be in $\operatorname{argmin} f$. By definition $f(x_k) + \frac{1}{2k}\|x_k\|^2 \leq f(a) + \frac{1}{2k}\|a\|^2$ $(*)$, thus

$$\|x_k\|^2 \leq 2k(f(a) - f(x_k)) + \|a\|^2 \leq \|a\|^2.$$

4/ One can simply pass to the limit in $(*)$ just above. One obtains the result.
Alternative proof : we have $\nabla f(x_k) + \frac{1}{k}x_k = 0$ hence if x^* is a limit point of x_k we have $\nabla f(x^*) = 0$ (we have seemingly used the continuity of the gradient here but it is not necessary, why ?). By convexity this means that x^* is a minimizer of f.

5/ S exactly defines the projection of 0 on the nonempty closed convex set $\operatorname{argmin} f$. Observe indeed that $\emptyset \neq \operatorname{argmin} f = [f \leq \min f]$. Thus by continuity and convexity of f, $\operatorname{argmin} f$ is closed and convex.

6/ f is twice differentiable as a composition of smooth functions. Its hessian (...) is given by A^TA which is positive semidefinite, indeed

$$\langle A^TAx, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2 \geq 0, \quad \forall x \in H.$$

The rest has already been done.

7/ By convexity : x minimizes f if and only if $\nabla f(x) = A^TAx - A^Tb = 0$. Thus $\operatorname{argmin} f \supset x^* + \operatorname{Ker} A$. Conversely if x minimizes f then $A^TAx = A^Tb$. Hence $A^TA(x - x^*) = 0$ therefore $\langle A^TA(x - v), x - v\rangle = 0$, i.e. $\|A(x - v)\|^2 = 0$.
Differentiating f_k gives $x_k = (\frac{1}{k}\operatorname{Id} + A^TA)^{-1}(A^TAb)$ where the invertibility of $\frac{1}{k}\operatorname{Id} + A^TA$ comes from the fact that A^TA is positive semidefinite.

Exercise 1

a) Just apply the definitions.
b) Observe that $n \geq 2$([1]). We establish that $-C_2 \subset C_2^*$. Let $(z_1, \ldots, z_n)$ in $-C_2$ and $(x_1, \ldots, x_n)$ in C_2. By Cauchy-Schwarz inequality

$$\begin{aligned} z_1x_1 + z_2x_2 + \ldots + z_nx_n &\leq z_1x_1 + \sqrt{z_2^2 + \ldots + z_n^2}\sqrt{x_2^2 + \ldots + x_n^2} \\ &\leq z_1x_1 - z_1x_1 \\ &= 0. \end{aligned}$$

We establish that $-C_2 \supset C_2^*$. Fix $(v_1, \ldots, v_n)$ in C_2^*. One has

$$v_1x_1 + v_2x_2 + \ldots v_nx_n \leq 0 \tag{1}$$

for all $(x_1, \ldots, x_n)$ in C_2. Choose x so that

$$x_1 = \sqrt{v_2^2 + \ldots + v_n^2}, \quad x_i = v_i \text{ for all } i \geq 2.$$

Using (1), one obtains

$$v_1\sqrt{v_2^2 + \ldots + v_n^2} \leq -v_2^2 - \ldots - v_n^2$$

which yields $v_1 \leq 0$ and $v_1^2 = v_2^2 + \ldots v_n^2$, i.e. $-v \in C_2$.

Exercise 2 a) In the interior we of course have *as always* : $N_{C_3}(x) = \{0\}$, $x \in \operatorname{int} C_3$. Assume thus x is on the boundary of C_3.

The most simple way at this stage of the classes is certainly to observe that

$$C_3 = \Big[||\cdot||^2 \leq 1\Big]$$

where $f = ||\cdot||^2$ is a convex smooth function. The Slater condition being clearly satisfied (take $x_0 = 0$), we thus have

$$N_{C_3}(x) = \mathbb{R}_+\nabla f(x) = \mathbb{R}_+(2x) = \mathbb{R}_+x.$$

A more down-to-earth but geometrical method is to consider curves of the form

$$t \to x_y(t) = (\cos t)\, x + (\sin t)\, y$$

where y is any normal vector to x such that $||y|| = 1$. This curve is drawn on the sphere and its velocity at zero is y. By definition

$$\lim_{t\to 0^+} \frac{1}{t}(x_y(t) - x) \in T_{C_3}(x)$$

that is $y \in T_{C_3}(x)$. Hence $T_{C_3}(x) \supset (\mathbb{R}x)^\perp$. This implies $N_{C_3}(x) \subset \mathbb{R}x$ and the result easily follows since $-x$ cannot be in the normal cone ($\langle -x, 0 - x\rangle = 1$).

b) It relies on Farkas lemma as done in Theorem 32 of the Static Optimization lecture notes.

1. Observe C_2 is the epigraph of a norm, it is thus convex.