Toulouse School of Economics,
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Optimization

The euclidean scalar product and the norm of the real vector space $\mathbb{R}^d$ ($d \in \mathbb{N}$) are respectively denoted by $\langle \cdot, \cdot \rangle$ and $|| \cdot ||$.
Let m, n be positive integers ; unless otherwise stated A denotes a matrix in $\mathbb{R}^{m\times n}$, while b, c are respectively vectors of $\mathbb{R}^m, \mathbb{R}^n$.
We recall that $\sup_{\emptyset} = -\infty$ and $\inf_{\emptyset} = +\infty$.

We also recall the following result :

Theorem 1 *The values of the primal and the dual of a feasible linear problem coincide.*

Exercice 1 Show that any linear program with inequality and equality constraints can be rewritten as

$$\begin{array}{ll} \min \langle c, x\rangle & \\ Ax \geq b & \text{(canonical form)}, \\ x \geq 0 & \end{array} \qquad \begin{array}{ll} \min \langle c, x\rangle & \\ Ax = b & \text{(standard form)}. \\ x \geq 0. & \end{array}$$

after a change of variable.

Exercice 2 Give the value and all the optimal solutions of the problem

$$\begin{array}{l} \min \langle c, x\rangle \\ \sum_{i=1}^n x_i = 1 \\ x \geq 0. \end{array}$$

Exercice 3 *(*)* Assume that the system $Ax = b, x \geq 0$ has a solution.
(a) Rewrite $Ax = b$ using the columns A_j of A.
(b) Show that there exists a solution which has at most m (strictly) positive components.
Hint. Take a solution x with $p > m$ positive components, and observe that the system $\{A_j : j \text{ such that } x_j \neq 0\}$ is linearly dependent. Use this remark to build a solution with at most $p-1$ positive components.

Exercice 4 ($\bullet$) Give a dual problem (nonnegativity/nonpositivity constraints, ie $x \geq 0/x \leq 0$ must not be "dualized") for the following programs : $\min\{\langle c, x\rangle : x \in C\}$ where C corresponds successively to the following sets $\{x : Ax \geq b\}$, $\{x : Ax = b, x \geq 0\}$, $\{x : Ax \geq b, x \leq 0\}$, $\{x : Ax \geq b, Lx = d, x \geq 0\}$(with $(L, d) \in \mathbb{R}^{p\times n} \times \mathbb{R}^p$).

Exercice 5 (•) Let S be a symmetric definite positive matrice (i.e. S is invertible and symmetric positive semidefinite). Consider the problem

$$(P) \quad \begin{array}{l} \min \langle Sx, x \rangle \\ Ax \leq b \end{array}$$

(a) Show that (P) has a unique solution iff (P) is feasible.
(b) Give a dual (D) for (P) and a sufficient condition to have val(P)=val(D).
(c) Write the optimality conditions for (P).

Exercice 6 Why are the following assertions equivalent ?
(i) $\exists x \in \mathbb{R}^n, x \geq 0, Ax = b$,
(ii) $\forall y \in \mathbb{R}^m, yA \geq 0 \Rightarrow \langle y, b \rangle \geq 0$.

Exercice 7 Prove Farkas lemma by using the duality theorem (i.e. Theorem 1).
Hint. Use the previous exercise. To prove (ii) $\Rightarrow$ (i), one could consider

$$\begin{array}{l} \max \langle y, b \rangle \\ yA \leq 0. \end{array}$$

Exercice 8 (•) Compute

$$\begin{array}{l} \min\{x_1 + 3x_2 + x_3 + 4x_4\} \\ x_1 + 3x_2 - x_3 + 4x_4 \geq 1, \\ 2x_1 + 6x_2 + x_3 + 8x_4 \geq 3, \\ x_1, \ldots, x_4 \geq 0. \end{array}$$

Exercice 9 Assume $m = n$, $A^t = A$ and suppose that the system $Ax = b, x \geq 0$ has a solution $\bar{x}$. Prove that the supremum in

$$\begin{array}{l} \sup \langle b, x \rangle \\ Ax \leq b \\ x \geq 0. \end{array}$$

is achieved at $\bar{x}$.

Exercice 10 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an arbitrary convex functions. Let S be the set of solutions of the *maximization* problem :

$$\sup\{f(x, y) : (x, y) \in [0, 1]^2\}.$$

Show that S contains one of the following points : $(0, 0); (0, 1); (1, 0); (1, 1)$.