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Optimization

- (•) Must-know question/answer/howtodo
- (*) Difficult
- (#) Optional

Convex sets

In the sequel H, $\langle\cdot,\cdot\rangle$ denotes a real Hilbert space.
When $X \subset H$, $I \subset \mathbb{R}$, we set $IX = I.X = \{tx : t \in I, x \in X\}$.
m and n denote positive integers.

1. (•) $\mathcal{E}, \mathcal{F}$ are two normed spaces.
(a) Let $f : \mathcal{E} \to \mathcal{F}$ be a continuous function. G is a closed subset of $\mathcal{F}$. Prove that $f^{-1}(G)$ is closed in $\mathcal{E}$.
(b) Ω is an open subset of $\mathcal{F}$. Prove that $f^{-1}(\Omega)$ is open in $\mathcal{E}$. (Observe that $\mathcal{E} \setminus f^{-1}(\Omega) = f^{-1}(\mathcal{F} \setminus \Omega)$ and set $G = \mathcal{F} \setminus \Omega$.)

2. (Convexity : miscellaneous) (a)(•) Let A be an $m \times n$ matrix and b in $\mathbb{R}^m$. Show that $\{x \in \mathbb{R}^n : Ax \le b\}$ is a closed convex set.
(b)(*) The space $E = C([0,1];\mathbb{R})$ is endowed with the sup norm $\|f\|_\infty = \max\{|f(x)| : x \in [0,1]\}$. Let g be in E an set $C = \{f \in E : f \ge g\}$. Show that C is closed and convex.
(c) Is it true that the set $\mathcal{S}_n$ of positive symmetric semidefinite matrices ([1]) is convex ?
Same question with the set of matrices of constant rank r ?

3. (Minkowski sum) Let A, B, C some subsets of $\mathbb{R}^n$.
(a) (Cancelation rule ?) Can we assert that $A + C = B + C$ implies $A = B$?
(b) Is it true that $\alpha C + \beta C = (\alpha + \beta)C$ for all $\alpha, \beta > 0$? What can be said if C is convex ?

4. (•) (Computing projections I)
(a) $e \in H \setminus \{0\}$. Set $A = \mathbb{R}e$. Show that A is a closed convex set and compute P_A.
(b) $c \in H \setminus \{0\}$. Set $B = \{x \in H : \langle x, c\rangle = 0\}$. Show that B is a closed convex set and compute P_B.
(c) Proceed now with the sets $A^+ = \mathbb{R}_+ e$ and $B^+ = \{x \in H : \langle x, c\rangle \ge 0\}$.

5. (#) (Computing projections II)
(a) Let $\bar{u} \in L^2((0,1);\mathbb{R})$ and $C = \{u \in L^2((0,1);\mathbb{R}) : u(x) \ge \bar{u}(x) \text{ a.e. on } (0,1)\}$. Show that C is closed and convex.
(b) Take $\bar{u} = 0$. Denote by P_C the projector onto C and give a formula for $P_C(u)$ where u is an arbitrary function in $L^2((0,1);\mathbb{R})$.

1. Recall that $S \in \mathcal{S}_n$ if $S^T = S$ and $x^T S x \ge 0$ for all vector x

6. (☺) ([2]) F is a nonempty closed subset (not necessarily convex) of the Euclidean space $\mathbb{R}^n, \langle , \cdot, \cdot \rangle$. (a) Let x be in $\mathbb{R}^n$. Can we say that there always exists in F a closest point to x ? In other words can we say that any x in $\mathbb{R}^n$ has at least one projection onto F ? ([3])
(b) Can we say that for any x close enough to F there exists a *unique* projection ?

7. (Projection on closed convex cones) We recall that $L \subset H$ is a cone if $\mathbb{R}_+ L \subset L$.
(1) Let $L \subset H$. Show that L is a convex cone if and only if $\mathbb{R}_+ L \subset L$ and $L + L \subset L$.
(2) Let L be a nonempty closed convex cone of H. Take x, z in H.
Show that $z = P_L(x)$ if and only if

$$\begin{cases} z \in L \\ \langle x - z, y \rangle \le 0, \ \forall y \in L \\ \langle x - z, z \rangle = 0. \end{cases}$$

8. (Interior and closure of convex sets)(*) Let C be a convex subset of H with nonempty interior. We admit the following fact

$$\forall x \in \overline{C}, \forall y \in \text{int } C,]x, y] \subset \text{int } C. \tag{1}$$

(a) Show that $\text{int } C = \text{int } \overline{C}$.
(b) Show that $\overline{\text{int } C} = \overline{C}$.
(c) (*) Prove (1).

9. (Supporting hyperplanes) Let C be a closed nonempty convex subset of H. Assume that $C \neq H$.
Establish the existence of $a \neq 0$ in H such that

$$(P_a) \qquad v_a := \sup\{\langle a, x \rangle : x \in C\} < +\infty,$$

where the supremum is *achieved.* Provide an example for which the sup is finite but not achieved.

10 (Extremality)**.** A point x of a convex set C is *extremal* if the following holds

$$\forall\, y, z \in C, \ \forall \lambda \in (0,1), \ \ x = \lambda y + (1 - \lambda) z \ \ \Rightarrow y = z = y.$$

The set of extremal points is denoted by $\mathcal{E}(C)$.
(1) Consider the vector space $E = \mathbb{R}^2$. What are the set of extremal points of the closed unit balls $B_{\|\cdot\|_1}$ and $B_{\|\cdot\|_\infty}$?
(2) Let B be the closed unit ball of H (an arbitrary Hilbert space). Show that $\mathcal{E}(B)$ coincides with the unit sphere – this property is sometimes called the smoothness/roundness of Hilbertian balls

11. (•)+(☺) Each of the following assertions are false. For each case give a counter-example and provide a nontrivial extra-assumption that makes the statement valid.
(a) The image of a closed convex set of $\mathbb{R}^n$ by a linear mapping is a closed convex set.
(b) Let $C \subset \mathbb{R}^n$ a convex set and $\mathcal{E}(C)$ its set of extremal points. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear application. Then $\mathcal{E}(L(C)) = L(\mathcal{E}(C))$.
(c) If C is a compact convex subset of $\mathbb{R}^n$ and x an extremal point of C, there exists an hyperplane H of $\mathbb{R}^n$ such that $H \cap C = \{x\}$ (just give a counter-example).
(d) Any two disjoint closed convex subsets of $\mathbb{R}^n$ can be *strongly* separated by an hyperplane.

2. Or possibly ☹ depending on your sense of humor
3. Formulate the question as an optimization problem...